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INEQUALITIES CONNECTING EIGENVALUES AND NONPRINCIPAL SUBDETERMINANTS

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ABSTRACT. The nonprincipal subdeterminants of a normal matrix satisfy certain quadratic identities. In this paper, these identities are used to obtain upper bounds on such subdeterminants in terms of elementary symmetric functions of the moduli of the eigenvalues. The same analysis yields lower bounds on the spread of a normal matrix and on the Hilbert norm of an arbitrary matrix.

1. STATEMENT OF RESULTS

Let $\lambda_1,\ldots,\lambda_n$ be n complex numbers. The totality of n-square normal matrices with these numbers as eigenvalues is the set of all matrices A of the form

$$A = U^* DU , \tag{1}$$

where U is unitary and $D = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$. It is well known [1, p. 237] that for a fixed integer m, $1 \leq m \leq n$, the totality $W_m(\lambda)$ of m-square principal subdeterminants of all A defined by (1) is a region in the plane contained in the convex polygon

$$P_m(\lambda) = \mathcal{H}\{\lambda_{\omega(1)} \cdots \lambda_{\omega(m)}, \quad \omega \in Q_{m,n}\} . \tag{2}$$

The notation in (2) is this: $Q_{m,n}$ is the set of all $\binom{n}{m}$ integer sequences ω having domain $\{1,\ldots,m\}$ and range contained in $\{1,\ldots,n\}$, and satisfying $\omega(1) < \omega(2) < \cdots < \omega(m)$; $\mathcal{H}$ denotes the convex hull of the indicated products. Thus

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$$W_m(\lambda) \subset P_m(\lambda) , \tag{3}$$

or in words, if A is a normal matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$ then any m-square principal subdeterminant of A lies in the polygon $P_m(\lambda)$. It is also known that in contrast to the case $m = 1$ when $W_1(\lambda)$ is the numerical range of any A, it is not generally the case for $1 < m < n - 1$ that $W_m(\lambda)$ is a convex set [4].

The situation for m-square nonprincipal subdeterminants is remarkably different. To fix the notation, let k,m be fixed integers, $1 \leq k < m < n$, and let $W_{k,m}(\lambda)$ denote the totality of m-square subdeterminants of the matrices A in (1) which have precisely k main-diagonal elements in common with A. More precisely,

$$W_{k,m}(\lambda) = \{\det A[\alpha|\beta] : \alpha,\beta \in Q_{m,n},\ |\operatorname{im}\alpha \cap \operatorname{im}\beta| = k,\ A \text{ defined by } (1)\} , \tag{4}$$

where $\operatorname{im}\alpha$ is the range of α and $A[\alpha|\beta]$ is the m-square submatrix of A lying in rows $\alpha(1),\ldots,\alpha(m)$ and columns $\beta(1),\ldots,\beta(m)$ of A. A slight modification of an argument found in [3, p. 220] shows that $W_{k,m}(\lambda)$ _is a closed circular disc centered at the origin_. Let $r_{k,m}(\lambda)$ denote the radius of this disc. Also let

$$E_m(|\lambda|) = E_m(|\lambda_1|,\ldots,|\lambda_n|)$$

denote the m-th elementary symmetric polynomial in $|\lambda_1|,\ldots,|\lambda_n|$, i.e.,

$$E_m(|\lambda|) = \sum_{\omega \in Q_{m,n}} \prod_{i=1}^{m} |\lambda_{\omega(i)}| .$$

The following is the main result of this paper.

THEOREM 1. _If_ $n \geq 4$, $m \geq 2$, _and_ $k \leq m - 2$, _then_

$$E_m(|\lambda|) \geq \begin{cases} 2(m - k + 1)r_{k,m}(\lambda) & \text{if } k < m - 2 , \\ 4r_{k,m}(\lambda) & \text{if } k = m - 2 . \end{cases} \tag{5}$$

In words, let A be a normal matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$, $n \geq 4$. Let B be an m-square submatrix of A having precisely k main-diagonal entries lying on the main diagonal of A. If $k \leq m - 2$, then

$$(6) \qquad |\det B| \le \begin{cases} \dfrac{E_m(|\lambda|)}{2(m-k+1)} & \text{if} \quad k < m-2 , \\ \dfrac{E_m(|\lambda|)}{4} & \text{if} \quad k = m-2 . \end{cases}$$

Recall that the <u>spread</u> of A [5, 6] is the number

$$s(A) = \max_{i,j} |\lambda_i - \lambda_j| .$$

We have the following result.

COROLLARY 1. *If* $2 < m < n$ *and the* m-*square submatrix* B *of* A *has no main-diagonal elements lying on the main diagonal of* A, *then*

$$(7) \qquad s(A) \ge \max \begin{cases} \sqrt{3}\left(2(m+1)\binom{n}{m}^{-1}\right)^{1/m} |\det B|^{1/m} , \\ 2\sqrt{6}\,(n(n-1))^{-1/2}\, |\det B|^{1/2} . \end{cases}$$

In the following corollary, A is an arbitrary n-square matrix ($n \ge 2$). Let d_m be the greatest m-square subdeterminant of A (in absolute value), and let $\|A\|$ be the Hilbert norm of A, i.e., the greatest singular value of A.

COROLLARY 2. *If* $2 < m \le n$, *then*

$$\|A\| \ge \max \begin{cases} \left(2(m+1)\binom{2n}{m}^{-1}\right)^{1/m} d_m^{1/m} , \\ 2(n(2n-1))^{-1/2}\, d_2^{1/2} . \end{cases}$$

The remainder of the paper is divided into two sections. In Section 2, a combinatorial lemma about sets of sequences is established to be used later in analyzing some consequences of the quadratic Plücker relations for subdeterminants. In Section 3, the proofs of the above results are given.

2. A COMBINATORIAL LEMMA

Let $\Gamma_{m,n}$ denote the set of all n^m integer sequences with domain

$\{1,\ldots,m\}$ and range $\{1,\ldots,n\}$. Let $\alpha,\beta \in Q_{m,n}$ $(1 \le m \le n)$, and let $s,t \in \{1,\ldots,m\}$. Define $\alpha[s,t:\beta]$ to be the sequence in $\Gamma_{m,n}$ obtained from α by replacing $\alpha(s)$ with $\beta(t)$:

$$\alpha[s,t:\beta] = (\alpha(1),\ldots,\alpha(s-1),\beta(t),\alpha(s+1),\ldots,\alpha(m)) .$$

Similarly, $\beta[t,s:\alpha]$ denotes the sequence in $\Gamma_{m,n}$ obtained from β by replacing $\beta(t)$ with $\alpha(s)$:

$$\beta[t,s:\alpha] = (\beta(1),\ldots,\beta(t-1),\alpha(s),\beta(t+1),\ldots,\beta(m)) .$$

As s and t vary over the set $\{1,\ldots,m\}$, they give rise to the following two lists of sequences in $\Gamma_{m,n}$:

$$\begin{array}{lcc}
 & \underline{\alpha[s,t:\beta] \text{ list}} & \underline{\beta[t,s:\alpha] \text{ list}} \\
\text{Block } s = 1 & \left\{\begin{array}{c}\alpha[1,1:\beta]\\ \vdots \\ \alpha[1,m:\beta]\end{array}\right. & \left.\begin{array}{c}\beta[1,1:\alpha]\\ \vdots \\ \beta[m,1:\alpha]\end{array}\right\} \\
 & \vdots & \vdots \\
\text{General Block } s & \left\{\begin{array}{c}\alpha[s,1:\beta]\\ \vdots \\ \alpha[s,m:\beta]\end{array}\right. & \left.\begin{array}{c}\beta[1,s:\alpha]\\ \vdots \\ \beta[m,s:\alpha]\end{array}\right\} \\
 & \vdots & \vdots \\
\text{Block } s = m & \left\{\begin{array}{c}\alpha[m,1:\beta]\\ \vdots \\ \alpha[m,m:\beta]\end{array}\right. & \left.\begin{array}{c}\beta[1,m:\alpha]\\ \vdots \\ \beta[m,m:\alpha]\end{array}\right\}
\end{array}$$

We shall refer to this array of sequences as "the twin lists." As indicated, the twin lists are arranged in m "blocks" (corresponding to $s = 1,\ldots,m$); each block has two columns (corresponding to α and β), each of which consists of m sequences (corresponding to $t = 1,\ldots,m$).

If $\gamma \in \Gamma_{m,n}$, we shall say that γ <u>appears in the twin lists</u> if the sequence $\gamma\sigma$ appears in the array for some permutation $\sigma \in S_m$.

LEMMA. *Suppose* $2 \leq m < n$, *and consider the sequences*

$$\alpha = (1,2,\dots,m) \in Q_{m,n}$$

and

$$\beta = (1,\dots,k,m+1,\dots,2m-k) \in Q_{m,n} ,$$

where $0 \leq k \leq m - 1$.

(i) *If* $k = m - 1$, *then* α *and* β *appear in every block in the twin lists for* α *and* β.

(ii) *If* $k \leq m - 2$, *then*

(a) *in each of blocks* $s = 1,\dots,k$ *in the twin lists, both* α *and* β *appear;*

(b) *in each of blocks* $s = k+1,\dots,m$ *in the twin lists, neither* α *nor* β *appears in rows* $k+1,\dots,m$;

(c) *in each of blocks* $s = k+1,\dots,m$ *in the twin lists, each of the first* k *sequences on the left involves repeated integers;*

(d) *in the totality of rows* $k+1,\dots,m$ *in blocks* $s = k+1,\dots,m$ *in the twin lists, no sequence appears more than once if* $k < m - 2$, *and the sequences which appear do so exactly twice if* $k = m - 2$.

Proof. As an introduction, let us write out the general block in the twin lists for α and β:

$$\begin{array}{ccc} & \alpha[s,t:\beta] & \beta[t,s:\alpha] \\ \begin{array}{c}\text{Block } s\\ (1\leq s\leq m)\end{array} & \left\{\begin{array}{l} (1,\dots,s-1,1,s+1,\dots,m) \\ (1,\dots,s-1,2,s+1,\dots,m) \\ \quad\vdots \\ (1,\dots,s-1,k,s+1,\dots,m) \\ (1,\dots,s-1,m+1,s+1,\dots,m) \\ \quad\vdots \\ (1,\dots,s-1,2m-k,s+1,\dots,m) \end{array}\right. & \begin{array}{l} (s,2,\dots,k,m+1,\dots,2m-k) \\ (1,s,\dots,k,m+1,\dots,2m-k) \\ \quad\vdots \\ (1,2,\dots,s,m+1,\dots,2m-k) \\ (1,2,\dots,k,s,\dots,2m-k) \\ \quad\vdots \\ (1,2,\dots,k,m+1,\dots,s) \end{array} \end{array}$$

(i) Suppose $k = m - 1$. Observe that if $s \leq m - 1$, then block s in the twin lists for α and β has the form:

$$\begin{array}{ll}
(1,\ldots,s-1,1,s+1,\ldots,m) & (s,2,\ldots,m-1,m+1) \\
\vdots & \vdots \\
(1,\ldots,s-1,s,s+1,\ldots,m) & (1,2,\ldots,s,\ldots,m-1,m+1) \\
\vdots & \vdots \\
(1,\ldots,s-1,m-1,s+1,\ldots,m) & (1,2,\ldots,s,m+1) \\
(1,\ldots,s-1,m+1,s+1,\ldots,m) & (1,2,\ldots,m-1,s) \;.
\end{array}$$

(Notice that since $k = m - 1$, we have $\beta = (1,\ldots,m-1,m+1)$.) Thus α appears as the s-th sequence on the left, and β appears as the s-th sequence on the right. Now block $s = m$ in the twin lists for α and β has the form:

$$\begin{array}{ll}
(1,\ldots,m-1,1) & (m,2,\ldots,m-1,m+1) \\
(1,\ldots,m-1,2) & (1,m,\ldots,m-1,m+1) \\
\vdots & \vdots \\
(1,\ldots,m-1,m-1) & (1,2,\ldots,m,m+1) \\
(1,\ldots,m-1,m+1) & (1,2,\ldots,m-1,m) \;,
\end{array}$$

and we see that α appears as the m-th sequence on the right, while β appears as the m-th sequence on the left. This establishes (i).

(ii) Suppose $k \leq m - 2$.

(a) If $s \in \{1,\ldots,k\}$, an inspection of block s in the twin lists immediately *shows* that α appears as the s-th sequence on the left, and β appears as the s-th sequence on the right.

(b) Let $s \in \{k+1,\ldots,m\}$. Then block s in the twin lists for α and β has the form:

$$\begin{array}{rll}
 & (1,\ldots,k,\ldots,\overset{\substack{\text{position } s\\ \downarrow}}{1},\ldots,m) & (s,\ldots,k,m+1,\ldots,2m-k) \\
 & \vdots & \vdots \\
 & (1,\ldots,k,\ldots,k,\ldots,m) & (1,\ldots,s,m+1,\ldots,2m-k) \\
(8) \quad \begin{array}{c}\text{rows}\\ k+1,\ldots,m\end{array} & \left\{\begin{array}{l} (1,\ldots,k,\ldots,m+1,\ldots,m) \\ (1,\ldots,k,\ldots,m+2,\ldots,m) \\ \vdots \\ (1,\ldots,k,\ldots,2m-k,\ldots,m) \end{array}\right. & \begin{array}{l} (1,\ldots,k,s,m+2,\ldots,2m-k) \\ (1,\ldots,k,m+1,s,\ldots,2m-k) \\ \vdots \\ (1,\ldots,k,m+1,\ldots,s) \;. \end{array}
\end{array}$$

Observe that each sequence in rows $k+1,\ldots,m$ in block s involves integers greater than m. Thus α does not appear in rows $k+1,\ldots,m$ in block s. Next, the $(k+1)$-st sequence on the right in block s does not involve $m+1$,. the $(k+2)$-nd sequence does not involve $m+2$, and so on until finally the m-th sequence does not involve $2m-k$. Thus β does not appear on the right in rows $k+1,\ldots,m$ in block s. Now if $s \leq m-1$, then every sequence on the left in rows $k+1,\ldots,m$ in block s involves m, and hence β does not appear on the left in these rows. If $s = m$, then rows $k+1,\ldots,m$ in block s have left-hand side of the form

$$\begin{array}{c} (1,\ldots,k,k+1,\ldots,m-1,m+1) \\ (1,\ldots,k,k+1,\ldots,m-1,m+2) \\ \vdots \\ (1,\ldots,k,k+1,\ldots,m-1,2m-k)\ , \end{array}$$

and each of these sequences involves $k+1$. But β does not involve $k+1$ since $k \leq m-2$, so again β does not appear on the left in rows $k+1,\ldots,m$ in block s. This completes the proof of (b).

(c) It is clear from the array (8) in the proof of (b) that if $s \in \{k+1,\ldots,m\}$, then each of the first k sequences on the left in block s involves repeated integers.

(d) Let us examine the array (8) in the proof of (b) both for a fixed s and for different values of $s \in \{k+1,\ldots,m\}$.

First, it is obvious that for a fixed $s \in \{k+1,\ldots,m\}$, the sequences on the left in block s are all distinct, as are the sequences on the right.

Next, let $s,s' \in \{k+1,\ldots,m\}$, $s \neq s'$, and observe that s does not occur in any sequence on the left in block s, whereas s does occur in every sequence on the left in block s'. Thus no sequence on the left in block s' appears on the left in block s. It follows from the preceding paragraph that in the totality of blocks $s = k+1,\ldots,m$ in the twin lists, no sequence appears more than once on the left.

Again, let $s,s' \in \{k+1,\ldots,m\}$, $s \neq s'$, and observe that s occurs in every sequence on the right in block s, whereas s does not occur in any sequence on the right in block s'. Thus no sequence on the right in block s' appears on the right in block s. As before, it follows that in the totality of blocks $s = k+1,\ldots,m$ in the twin lists, no sequence appears more than once on the right.

If $s \in \{k+1,\ldots,m-1\}$, then each sequence on the left in block s involves m and hence does not appear on the right in block s' for any $s' \in \{k+1,\ldots,m-1\}$. Also, each sequence on the right in block $s = m$ involves m and hence does not appear on the left in block m.

Now suppose $k < m - 2$. We wish to show that in the totality of rows $k+1,\ldots,m$ in blocks $s = k+1,\ldots,m$ in the twin lists, no sequence appears more than once. By the above observations, we need verify only that no sequence on the left in rows $k+1,\ldots,m$ in blocks $s = k+1,\ldots,m-1$ appears on the right in rows $k+1,\ldots,m$ in block m, and that no sequence on the left in rows $k+1,\ldots,m$ in block m appears on the right in rows $k+1,\ldots,m$ in blocks $s = k+1,\ldots,m-1$. We reproduce the twin lists for α and β, omitting blocks $1,\ldots,k$ and rows $1,\ldots,k$ in each of the blocks $s = k+1,\ldots,m$:

$$\begin{array}{ll} \begin{array}{l}\text{Block } s,\\ k+1 \le s \le m-2;\\ t = k+1,\ldots,m\end{array} & \left\{\begin{array}{ll} (1,\ldots,k,\ldots,m+1,\ldots,m-1,m) & (1,\ldots,k,s,m+2,\ldots,2m-k) \\ \vdots & \vdots \\ (1,\ldots,k,\ldots,2m-k,\ldots,m-1,m) & (1,\ldots,k,m+1,m+2,\ldots,s) \end{array}\right. \\ \\ \begin{array}{l}\text{Block } s = m-1;\\ t = k+1,\ldots,m\end{array} & \left\{\begin{array}{ll} (1,\ldots,k,\ldots,m-2,m+1,m) & (1,\ldots,k,m-1,m+2,\ldots,2m-k) \\ \vdots & \vdots \\ (1,\ldots,k,\ldots,m-2,2m-k,m) & (1,\ldots,k,m+1,m+2,\ldots,m-1) \end{array}\right. \\ \\ \begin{array}{l}\text{Block } s = m;\\ t = k+1,\ldots,m\end{array} & \left\{\begin{array}{ll} (1,\ldots,k,\ldots,m-2,m-1,m+1) & (1,\ldots,k,m,m+2,\ldots,2m-k) \\ \vdots & \vdots \\ (1,\ldots,k,\ldots,m-2,m-1,2m-k) & (1,\ldots,k,m+1,m+2,\ldots,m) . \end{array}\right. \end{array}$$

Inspection of this array shows that each sequence on the left in rows $k+1,\ldots,m$ in blocks $s = k+1,\ldots,m-2$ involves $m - 1$ and hence does not appear on the right in block m; each sequence on the left in rows $k+1,\ldots,m$ in block $m - 1$ involves $m - 2$ and hence does not appear on the right in block m (since $k < m - 2$); each sequence on the left in rows $k+1,\ldots,m$ in block m involves $m - 1$ and hence does not appear on the right in rows $k+1,\ldots,m$ in blocks $s = k+1,\ldots,m-2$; each sequence on the left in rows $k+1,\ldots,m$ in block m involves $m - 2$ and hence does not appear on the right in rows $k+1,\ldots,m$ in block $m - 1$. This completes the required verification and establishes the assertion in (d) for the case $k < m - 2$.

Finally, suppose $k = m - 2$. Then if we consider the totality of rows $m-1\ (=k+1), m$ in blocks $s = m-1, m$ in the twin lists for α and β,

$$\begin{array}{ll} \text{Block } s = m-1; & \begin{cases} (1,\dots,m-2,m+1,m) & (1,\dots,m-2,m-1,m+2) \\ (1,\dots,m-2,m+2,m) & (1,\dots,m-2,m+1,m-1) \end{cases} \\ t = m-1,m & \end{array}$$

$$\begin{array}{ll} \text{Block } s = m & \begin{cases} (1,\dots,m-2,m-1,m+1) & (1,\dots,m-2,m,m+2) \\ (1,\dots,m-2,m-1,m+2) & (1,\dots,m-2,m+1,m)\ , \end{cases} \\ t = m-1,m & \end{array}$$

we see immediately that every sequence which appears does so exactly twice. This establishes the assertion in (d) for the case $k = m - 2$. □

3. PROOFS

Proof of Theorem 1. We shall prove the equivalent statement that if $U \in U_n(\mathbb{C})$ is any unitary matrix, then

$$|\det(U^*AU)[\alpha|\beta]| \le \begin{cases} \dfrac{E_m(|\lambda_1|,\dots,|\lambda_n|)}{2(m-k+1)} & \text{if} \quad k < m-2\ , \\ \frac{1}{4} E_m(|\lambda_1|,\dots,|\lambda_n|) & \text{if} \quad k = m-2\ . \end{cases}$$

We begin by making the following two reductions. First, we may assume that A is diagonal,

$$A = \mathrm{diag}(\lambda_1,\dots,\lambda_n)\ .$$

Second, by effecting an appropriate permutation similarity transformation on the matrix U^*AU, we may assume

$$\alpha = (1,2,\dots,m) \quad \text{and} \quad \beta = (1,\dots,k,m+1,\dots,2m-k)\ .$$

Fix a matrix $U \in U_n(\mathbb{C})$, and let

$$\Delta = |\det(U^*AU)[\alpha|\beta]| = |C_m(U^*AU)_{\alpha,\beta}|\ ,$$

where $C_m(X)$ is the m-th compound matrix [1, p. 127]. For each $\nu \in Q_{m,n}$, let

$$p_\nu(\gamma)^{\cdot} = \det U[\nu|\gamma] \qquad \gamma \in \Gamma_{m,n} .$$

We have

$$\begin{aligned} C_m(U^*AU)_{\alpha,\beta} &= \sum_{\nu,\mu\in Q_{m,n}} C_m(U^*)_{\alpha,\nu}\, C_m(A)_{\nu,\mu}\, C_m(U)_{\mu,\beta} \\ &= \sum_{\nu\in Q_{m,n}} \overline{\det U[\nu|\alpha]} \det A[\nu|\nu] \det U[\nu|\beta] \\ &= \sum_{\nu\in Q_{m,n}} \overline{p_\nu(\alpha)}\, \lambda_\nu p_\nu(\beta) , \end{aligned}$$

where $\lambda_\nu = \lambda_{\nu(1)} \cdots \lambda_{\nu(m)}$. Therefore,

$$\Delta \le \sum_{\nu\in Q_{m,n}} |\lambda_\nu||p_\nu(\alpha)||p_\nu(\beta)| . \tag{9}$$

Now the quadratic Plücker relations [2, p. 10] imply that for each $\nu \in Q_{m,n}$ and any $s \in \{1,\dots,m\}$,

$$p_\nu(\alpha)p_\nu(\beta) = \sum_{t=1}^{m} p_\nu(\alpha[s,t:\beta])\, p_\nu(\beta[t,s:\alpha]) . \tag{10}$$

Taking absolute values in (10), applying the triangle inequality, and summing both sides on $s = k+1,\dots,m$, we obtain, for each $\nu \in Q_{m,n}$,

$$|p_\nu(\alpha)||p_\nu(\beta)| \le \frac{1}{m-k} \sum_{s=k+1}^{m} \sum_{t=1}^{m} |p_\nu(\alpha[s,t:\beta])||p_\nu(\beta[t,s:\alpha])| . \tag{11}$$

Combining (9) and (11) yields

$$\Delta \le \frac{1}{m-k} \sum_{\nu\in Q_{m,n}} |\lambda_\nu| \left\{ \sum_{s=k+1}^{m} \sum_{t=1}^{m} |p_\nu(\alpha[s,t:\beta])||p_\nu(\beta[t,s:\alpha])| \right\},$$

and it follows from part (ii)(c) of the lemma in Section 2 and the arithmetic-geometric mean inequality that

$$\begin{aligned} \Delta &\le \frac{1}{m-k} \sum_{\nu\in Q_{m,n}} |\lambda_\nu| \left\{ \sum_{s=k+1}^{m} \sum_{t=k+1}^{m} |p_\nu(\alpha[s,t:\beta])||p_\nu(\beta[t,s:\alpha])| \right\} \\ &\le \frac{1}{2(m-k)} \sum_{\nu\in Q_{m,n}} |\lambda_\nu| \left\{ \sum_{s=k+1}^{m} \sum_{t=k+1}^{m} |p_\nu(\alpha[s,t:\beta])|^2 + |p_\nu(\beta[t,s:\alpha])|^2 \right\}. \end{aligned} \tag{12}$$

Let us denote the double summation inside the brackets in the second inequality in (12) by $\{\Sigma_{s,t}\}$.

Since U is a unitary matrix, so is the m-th compound $C_m(U)$. Hence for each $\nu \in Q_{m,n}$, the sum of the squares of the moduli of the elements $p_\nu(\omega)$, $\omega \in Q_{m,n}$, in row ν of $C_m(U)$ is 1. It follows from parts (ii) (b), (d) of the lemma that

$$\left\{\sum_{s,t}\right\} + |p_\nu(\alpha)|^2 + |p_\nu(\beta)|^2 \le 1 \qquad \text{if} \quad k < m - 2 , \tag{13}$$

$$\frac{1}{2}\left\{\sum_{s,t}\right\} + |p_\nu(\alpha)|^2 + |p_\nu(\beta)|^2 \le 1 \qquad \text{if} \quad k = m - 2 . \tag{14}$$

The remainder of the argument consists of a calculation performed in two cases.

<u>Case I</u>: $k < m - 2$. From (12) and (13) we conclude that

$$\begin{aligned}
\Delta &\le \frac{1}{2(m-k)} \sum_{\nu \in Q_{m,n}} |\lambda_\nu|(1 - (|p_\nu(\alpha)|^2 + |p_\nu(\beta)|^2)) \\
&= \frac{1}{2(m-k)} \left[\sum_{\nu \in Q_{m,n}} |\lambda_\nu| - \sum_{\nu \in Q_{m,n}} |\lambda_\nu|(|p_\nu(\alpha)|^2 + |p_\nu(\beta)|^2)\right] \\
&\le \frac{1}{2(m-k)} \left[\sum_{\nu \in Q_{m,n}} |\lambda_\nu| - 2 \sum_{\nu \in Q_{m,n}} |\lambda_\nu||p_\nu(\alpha)||p_\nu(\beta)|\right] \\
&\le \frac{1}{2(m-k)} \sum_{\nu \in Q_{m,n}} |\lambda_\nu| - \frac{\Delta}{m-k} \qquad \text{(by (9))} .
\end{aligned}$$

Therefore

$$\Delta + \frac{\Delta}{m-k} \le \frac{1}{2(m-k)} \sum_{\nu \in Q_{m,n}} |\lambda_\nu| ,$$

so that

$$\Delta \le \frac{1}{2(m-k+1)} \sum_{\nu \in Q_{m,n}} |\lambda_\nu| = \frac{E_m(|\lambda_1|,\ldots,|\lambda_n|)}{2(m-k+1)} .$$

<u>Case II</u>: $k = m - 2$. From (12) (with $k = m - 2$) and (14) we conclude that

$$\begin{aligned}
\Delta &\le \frac{1}{4} \sum_{\nu \in Q_{m,n}} |\lambda_\nu| \, 2(1 - (|p_\nu(\alpha)|^2 + |p_\nu(\beta)|^2)) \\
&= \frac{1}{2} \left[\sum_{\nu \in Q_{m,n}} |\lambda_\nu| - \sum_{\nu \in Q_{m,n}} |\lambda_\nu|(|p_\nu(\alpha)|^2 + |p_\nu(\beta)|^2)\right] \\
&\le \frac{1}{2} \left[\sum_{\nu \in Q_{m,n}} |\lambda_\nu| - 2 \sum_{\nu \in Q_{m,n}} |\lambda_\nu||p_\nu(\alpha)||p_\nu(\beta)|\right]
\end{aligned}$$

$$\leq \frac{1}{2} \sum_{\nu \in Q_{m,n}} |\lambda_\nu| - \Delta \qquad \text{(by (9))} .$$

Therefore

$$2\Delta \leq \frac{1}{2} \sum_{\nu \in Q_{m,n}} |\lambda_\nu| ,$$

so that

$$\Delta \leq \frac{1}{4} \sum_{\nu \in Q_{m,n}} |\lambda_\nu| = \frac{1}{4} E_m(|\lambda_1|, \ldots, |\lambda_n|) .$$

Since $\Delta = |\det(U^*AU)[\alpha|\beta]|$, this completes the proof of the theorem. □

<u>Proof of Corollary 1</u>. Assume first that $2 < m < n$, and let $\alpha, \beta \in Q_{m,n}$ be sequences such that

$$\operatorname{im} \alpha \cap \operatorname{im} \beta = \emptyset, \qquad \text{so that} \qquad B = A[\alpha|\beta] .$$

For any $t \in \mathbf{C}$, $A - tI_n \in M_n(\mathbf{C})$ is a normal matrix with eigenvalues $\lambda_1 - t, \ldots, \lambda_n - t$, and since $\operatorname{im} \alpha \cap \operatorname{im} \beta = \emptyset$ we have

$$(A - tI_n)[\alpha|\beta] = A[\alpha|\beta] .$$

It follows by Theorem 1 that

$$|\det A[\alpha|\beta]| = |\det(A - tI_n)[\alpha|\beta]|$$

$$\leq \frac{E_m(|\lambda_1 - t|, \ldots, |\lambda_n - t|)}{2(m+1)} \leq \frac{\binom{n}{m}\left(\max_{1 \leq i \leq n} |\lambda_i - t|\right)^m}{2(m+1)} .$$

Since this is true for each $t \in \mathbf{C}$, we have

$$|\det A[\alpha|\beta]| \leq \frac{\binom{n}{m}\left(\min_{t \in \mathbf{C}} \max_{1 \leq i \leq n} |\lambda_i - t|\right)^m}{2(m+1)} . \tag{15}$$

Now it is known [5] that any n points $\lambda_1, \ldots, \lambda_n$ in $\mathbf{C}$ are contained in a disc of radius

$$\frac{\max_{1 \leq i,\, j \leq n} |\lambda_i - \lambda_j|}{\sqrt{3}} = \frac{s(A)}{\sqrt{3}} .$$

If t_0 is the center of this disk, then certainly

$$\max_{1\le i\le n} |\lambda_i - t_0| \le \frac{s(A)}{\sqrt{3}} ,$$

and hence

(16) $$\min_{t\in C} \max_{1\le i\le n} |\lambda_i - t| \le \frac{s(A)}{\sqrt{3}} .$$

The inequalities (15) and (16) together imply that

$$|\det A[\alpha|\beta]| \le \frac{\binom{n}{m}\left(\frac{s(A)}{\sqrt{3}}\right)^m}{2(m+1)} ,$$

whence

$$s(A) \ge \sqrt{3}\left(\frac{2(m+1)}{\binom{n}{m}}\right)^{1/m} |\det A[\alpha|\beta]|^{1/m} .$$

Since the sequences $\alpha,\beta \in Q_{m,n}$ were arbitrarily chosen subject to the condition $\operatorname{im}\alpha \cap \operatorname{im}\beta = \emptyset$, we conclude that

$$s(A) \ge \sqrt{3}\left[\frac{2(m+1)}{\binom{n}{m}}\right]^{1/m} \max_{\substack{\alpha,\beta\in Q_{m,n}\\ \operatorname{im}\alpha\cap\operatorname{im}\beta=\emptyset}} |\det A[\alpha|\beta]|^{1/m}$$

whenever $2 < m < n$, and the inequality for the first expression on the right in (7) is established.

The proof of the inequality for the second expression on the right in (7) is virtually identical, the sole modification being that the application of Theorem 1 involves the case $m = 2$ rather than $m > 2$; this has the effect of replacing the constant $2(m+1)$ by 4 throughout. □

Note that if A is hermitian (or skew hermitian), it is clear that

(17) $$\min_{t\in C} \max_{1\le i\le n} |\lambda_i - t| = \frac{s(A)}{2} .$$

Using (17) in place of (16) in the proofs of (7), we obtain the following specialization of Corollary 1.

<u>If</u> A <u>is hermitian or skew hermitian and the</u> m<u>-square submatrix</u> B <u>of</u> A <u>has no main-diagonal elements lying on the main diagonal of</u> A, <u>then</u>

(18) $$s(A) \ge 2\left(2(m+1)\binom{n}{m}^{-1}\right)^{1/m} |\det B|^{1/m}, \quad 2 < m < n ,$$

and

(19) $$s(A) \geq 4\sqrt{2}(n(n-1))^{-1/2} |\det B|^{1/2} .$$

Proof of Corollary 2. The matrix

$$\tilde{A} = \left[\begin{array}{c|c} 0 & A \\ \hline A^* & 0 \end{array}\right] \in M_{2n}(C)$$

is hermitian with eigenvalues $\pm\alpha_1,\ldots,\pm\alpha_n$, where

$$\alpha_1 = \|A\| \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$$

are the singular values of A. Applying the inequality (18) to $\tilde{A}$, we see that if $2 < m \leq n$ then

$$s(\tilde{A}) \geq 2\left[\frac{2(m+1)}{\binom{2n}{m}}\right]^{1/m} \max_{\substack{\gamma,\omega \in Q_{m,2n} \\ \mathrm{im}\,\gamma \cap \mathrm{im}\,\omega = \emptyset}} |\det \tilde{A}[\gamma|\omega]|^{1/m}$$

$$\geq 2\left[\frac{2(m+1)}{\binom{2n}{m}}\right]^{1/m} \max_{\alpha,\beta \in Q_{m,n}} |\det A[\alpha|\beta]|^{1/m} .$$

Since $s(\tilde{A}) = 2\alpha_1$, the inequality for the first expression on the right in Corollary 2 follows. In the same way, application of (19) to $\tilde{A}$ yields the inequality for the second expression on the right in Corollary 2. □

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